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1. Introduction. A first order stochastic differential equation is any equation which can be expressed symbolically in the form

$$dy(t) = m[t, y(t)]dt + \sigma[t, y(t)]dz(t); \tag{1. 1}$$

m and σ are called the drift and diffusion coefficients and $z(\cdot)$ is usually a Brownian motion process. If $m[t, x] \equiv m_0 x$ and $\sigma[t, x] \equiv \sigma_0$, where m_0 and σ_0 are constants, then this equation is called the Langevin equation, and its importance has been recognized for some time in many problems of physics and engineering. The rigorous interpretation and the development of the corresponding theory of the y-process, with the Itô-Doob approach to stochastic integrals, comprises part of diffusion theory (i.e. the theory of Markov processes with continuous sample paths) and is treated in detail in the recent books of Dynkin [5] and Skorokhod [18].

The following related problem has received little attention thus far. It concerns the simple harmonic oscillator driven by a Brownian disturbance (i.e. "white noise"), given by the symbolic stochastic equation

$$dy'(t) + 2\alpha y'(t)dt + \beta^2 y(t)dt = dz(t),$$

where y' denotes the sample derivative of the y-process describing the position of the particle, the z-process again being Brownian motion. This type of equation leads naturally to non-linear extensions of the form

$$dy'(t) = m[t, y(t), y'(t)]\, dt + \sigma[t, y(t), y'(t)]dz(t). \tag{1. 2}$$

In this paper we shall study the stochastic processes satisfying equation (1. 2). The solution process, i.e. the y-process, will be called a *second order Itô process* following the terminology introduced by Borchers [2].

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The necessary preliminaries are presented in the next two sections. Then the oscillation of the sample paths of the y'-process is studied with the aid of some general results on stochastic integrals, which are then applied to the study of symmetrized stochastic integrals. Next, a local and a global comparison theorem for the sample functions of second order Itô processes are obtained; these results justify the use of the term "drift coefficient" for m. The sample paths of the y'-process are shown to be directly related to the sample paths of a Brownian motion process in case σ is non-random. Finally, the asymptotic sample function behavior of second order Itô processes is studied in the stationary case with the aid of martingale theory.

2. Stochastic differential equations. Let $(\Omega, \mathscr{F}, P)$ denote a fixed probability space on which all random variables are defined. $E\{x\}$ denotes the expectation of a random variable x. $R^+ = [0, \infty)$ and R^n is n-dimensional Euclidean space. The "ω" variable will usually be suppressed when writing random variables so that, for example, the stochastic process $\{x(t, \cdot), t \in R^+\}$ will be written as $\{x(t),\ t \in R^+\}$. Two random variables x and y are identified if $P\{\omega\colon\ x(\omega) = y(\omega)\} = 1$. All stochastic processes considered below can and will be assumed separable (cf. Doob [4, p. 57]).

A stochastic differential equation is an equation of the form

$$dY(t) = M[t, Y(t)]dt + S[t, Y(t)]dZ(t),$$

the rigorous interpretation of which is obtained, via stochastic integrals, by writing it in the integrated form

$$Y(t) = Y(t_0) + \int_{t_0}^{t} M[s, Y(s)]ds + \int_{t_0}^{t} S[s, Y(s)]dZ(s). \tag{2. 1}$$

Here $\{Z(t),\ t \in R^+\}$ is a normalized m-dimensional Brownian motion (or Wiener process): $Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_m(t) \end{pmatrix}$, and $\{z_1(t),\ t \in R^+\}, \cdots, \{z_m(t),\ t \in R^+\}$ are m independent one-dimensional Brownian motion processes, each with unit variance parameter. $M(\cdot, \cdot)$ maps $R^+ \times R^n$ into R^n, and $S(\cdot, \cdot)$ maps $R^+ \times R^n$ into the set of all real $n \times m$ matrices. If the i-th component of $M(t, x)$ is $m_i(t, x)$, if the i-th component of $Y(t)$ is $y_i(t)$, and if the ij-th component of $S(t, x)$ is $\sigma_{ij}(t, x)$, then the vector equation (2. 1) can be written in the form

$$y_i(t) = y_i(t_0) + \int_{t_0}^{t} m_i[s, Y(s)]\, ds + \sum_{j=1}^{m} \int_{t_0}^{t} \sigma_{ij}[s, Y(s)] dz_j(s), \tag{2. 2}$$

$i = 1, \cdots, n$. The last term in this equation is a sum of (Itô) stochastic integrals. Equation (2. 1) or (2. 2) is to be solved for $t \geq t_0$. The following assumptions are made:

(A1) Each $m_i(\cdot, \cdot)$ and each $\sigma_{ij}(\cdot, \cdot)$ are Baire functions.

(A2) For each $x \in R^n$, each $t \in R^+$, and each i, j,

$$\int_0^t (|m_i(s, x)|^2 + |\sigma_{ij}(s, x)|^2) ds < \infty.$$

(A3) For each $x \in R^n$ and each $t \in R^+$, $S(t, x)$ is a non-negative definite matrix.

(A4) For each $T > 0$ there is a constant $K(T)$ such that if $t \in [0, T]$ and $x, y \in R^n$ then

$$|m_i(t, x) - m_i(t, y)|, \quad |\sigma_{ij}(t, x) - \sigma_{ij}(t, y)| \leq K(T) |x - y|$$

for $i = 1, \cdots, n$, $j = 1, \cdots, m$, where $|x| = (\sum_{k=1}^{n} x_k^2)^{1/2}$ is the Euclidean norm of $x \in R^n$.

(A5) For $i = 1, \cdots, n$, the random variable $y_i(t_0)$ is square integrable and is independent of the increments $\{Z(v) - Z(u),\ t_0 \leq u \leq v < \infty\}$.

Then there exists an essentially unique (vector) stochastic process $\{Y(t),\ t \geq t_0\}$ such that equation (2. 1) (or equivalently (2. 2)) holds with probability 1 for each $t \geq t_0$. The "essential uniqueness" means that if $\{X(t),\ t \geq t_0\}$ is another solution process, then $P\{\omega\colon Y(t, \omega) = X(t, \omega)\} = 1$ for each $t \geq t_0$. $Y(t)$ is measurable relative to $\mathscr{F}(t_0, t)$, the σ-field generated by the increments $\{Z(v) - Z(u),\ t_0 \leq u \leq v \leq t\}$ and $Y(t_0)$. Moreover, $\{Y(t),\ t \geq t_0\}$ is a (vector) Markov process. The transition function of the Y-process is stationary if $M(t, x) \equiv M(x)$ and $S(t, x) \equiv S(x)$ are independent of t. With probability 1, the sample functions $Y(\cdot, \omega)$ are continuous functions from R^+ to R^n. For proofs see Dynkin [5] or Skorokhod [18]. The original study is due to Itô [14].

3. Second order Itô processes. These processes were introduced by Borchers [2]. A second order Itô process is a stochastic process $\{y(t),\ t \in R^+\}$ which satisfies the stochastic differential equation

$$dy'(t) = m[t, y(t), y'(t)]dt + \sigma[t, y(t), y'(t)]dz(t).$$

Here $\{z(t),\ t \in R^+\}$ is a Brownian motion process with unit variance parameter. The exact meaning of the above equation is

$$\begin{aligned} y(t) &= y(0) + \int_0^t y'(s)ds \\ y'(t) &= y'(0) + \int_0^t m[s, y(s), y'(s)]ds + \int_0^t \sigma[s, y(s), y'(s)]dz(s), \end{aligned} \tag{3. 1}$$

the last integral being a stochastic integral. Equations (3. 1) are to hold with probability 1 so that $\{y'(t),\ t \in R^+\}$ is the sample derivative process of $\{y(t),\ t \in R^+\}$. The following conditions are assumed to hold:

(B1) $m(\cdot, \cdot), \sigma(\cdot, \cdot)$: $R^+ \times R^2 \longrightarrow R^1$ are Baire functions.

(B2) For each $T > 0$ there is a constant $K(T)$ such that if $t \in [0, T]$ and if $x, y \in R^2$ then

$$0 \leq |m(t, x)|,\ \sigma(t, x) \leq K(T)(1 + |x|^2)^{1/2},$$
$$|m(t, x) - m(t, y)|,\ |\sigma(t, x) - \sigma(t, y)| \leq K(T)|x - y|.$$

(B3) $y(0)$, $y'(0)$ are square integrable random variables which are independent of all the increments of the z-process.

Equations (3. 1) can be written as a vector stochastic equation of the form (2. 1), where

$$Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix},\quad M(t, \xi, \eta) = \begin{pmatrix} \eta \\ m(t, \xi, \eta) \end{pmatrix},$$
$$S(t, \xi, \eta) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(t, \xi, \eta) \end{pmatrix},\quad Z(t) = \begin{pmatrix} z_0(t) \\ z(t) \end{pmatrix};$$

here $\{z_0(t),\ t \in R^+\}$ is a (dummy) Brownian motion with unit variance parameter independent of the z-process and of $Y(0)$. For notational simplicity we have taken as the initial time $t_0 = 0$. The conditions (B1)—(B3) imply the conditions (A1)—(A5), so there is an essentially unique solution process $\left\{Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix},\ t \in R^+\right\}$ which is a (vector) Markov process, and, with probability 1, $y(\cdot, \omega)$ and $y'(\cdot, \omega)$ are continuous on R^+. Moreover, $y'(t)$ is the "mean square" (or strong $L^2(\Omega, \mathscr{F}, P)$) derivative of $y(t)$, i.e.

$$\lim_{h \to 0} E\{|h^{-1}(y(t+h) - y(t)) - y'(t)|^2\} = 0,\quad t \in R^+;$$

in addition, $\{y'(t),\ t \in R^+\}$ is separable and measurable, and $\{y'(t),\ 0 \leq t \leq T\}$

is uniformly integrable for each $T>0$. This was proved by Borchers [2, Theorem 2. 1]. The y'-process, which is the derived process of $\{y(t), t\in R^+\}$ in both the strong (mean-square) and pointwise senses, can thus be called the *stochastic derivative* of the y-process (following Borchers [2]).

4. Variation of the sample paths. The main results of this section are valid for general diffusion processes which are solutions of equation (2. 1) (with $t_0=0$ for convenience). Thus these results also apply to second order Itô processes (see Corollary 4. 4 for example). Applications will then be made to symmetrized stochastic integrals.

Recall that for $t\in R^+$, $\mathscr{F}(0,t)$ is the σ-field generated by the random vectors $\{Z(v)-Z(u),\ 0\leq u\leq v\leq t\}$ and $Y(0)$. We shall consider stochastic processes $\{c(t),\ t\in R^+\}$ for which the following three conditions are satisfied:

(C1) $c(\cdot,\cdot)$ is a measurable function on $R^+\times\Omega$ (thus $\{c(t),\ t\in R^+\}$ is a measurable stochastic process).

(C2) $c(t)$ is $\mathscr{F}(0,t)$ measurable for each $t\in R^+$.

(C3) $P\left\{\omega\colon \int_0^T c^2(t,\omega)dt<\infty\right\}=1$ for each $T>0$.

Let $\{\alpha_i(t),\ t\in R^+\}$, $\{\beta_{ij}(t),\ t\in R^+\}$, $1\leq i\leq n$, $1\leq j\leq m$, be stochastic processes for which (C1)–(C3) hold. Let $0\leq a<b<\infty$. Set

$$x_i(t)=\int_a^t\alpha_i(s)ds+\sum_{j=1}^m\int_a^t\beta_{ij}(s)dz_j(s),\quad t\geq a,\ \ i=1,\cdots,n.$$

Suppose that

$$\pi\colon a=t_0<t_1<\cdots<t_k=b \tag{4. 1}$$

is a partition of $[a,b]$ with norm $|\pi|=\max_{0\leq p\leq k-1}(t_{p+1}-t_p)$.

THEOREM 4. 1. *Let the α_i- and β_{ij}-processes satisfy* (C1)-(C3) *and let the x_i-processes be defined as above. For $i,\ j=1,\cdots,n$, the limit in probability of*

$$\sum_{p=0}^{k-1}(x_i(t_{p+1})-x_i(t_p))(x_j(t_{p+1})-x_j(t_p))$$

as $|\pi|$ tends to zero exists and equals

$$\sum_{k=1}^m\int_a^b\beta_{ik}(s)\beta_{jk}(s)ds.$$

In particular, the limit in probability of

$$\sum_{p=0}^{2^k-1} |x_i(a+2^{-k}(b-a)(p+1)) - x_i(a+2^{-k}(b-a)p)|^2$$

as $k \longrightarrow \infty$ *exists and equals*

$$\sum_{k=1}^{m} \int_a^b \beta_{ik}^2(s)ds.$$

Proof. Observe that the second statement in the theorem is a special case of the first (with $i = j$). Let π be a partition of $[a, b]$ as in (4. 1). Then by Itô's formula [13, Lemma 5, p. 62],

$$\begin{aligned}
&(x_i(t_{p+1}) - x_i(t_p))(x_j(t_{p+1}) - x_j(t_p)) \\
&= \int_{t_p}^{t_{p+1}} \{[x_i(\tau) - x_i(\lambda_\pi(\tau))]\alpha_j(\tau) + [x_j(\tau) - x_j(\lambda_\pi(\tau))]\alpha_i(\tau)\} d\tau \\
&+ \sum_{r=1}^{m} \int_{t_p}^{t_{p+1}} \{[x_i(\tau) - x_i(\lambda_\pi(\tau))]\beta_{jr}(\tau) + [x_j(\tau) - x_j(\lambda_\pi(\tau))]\beta_{ir}(\tau)\} dz_r(\tau) \\
&+ \sum_{r=1}^{m} \int_{t_p}^{t_{p+1}} \beta_{ir}(\tau)\beta_{jr}(\tau) d\tau\,,
\end{aligned}$$

where $\lambda_\pi(\tau) = \max\{t_j\colon\ t_j < \tau\}$. Summing from $p = 0$ to $p = k-1$ we get

$$\begin{aligned}
&\sum_{p=o}^{k-1} (x_i(t_{p+1}) - x_i(t_p))(x_j(t_{p+1}) - x_j(t_p)) - \sum_{r=1}^{m} \int_a^b \beta_{ir}(\tau)\beta_{jr}(\tau) d\tau \\
&= \int_a^b \{[x_i(\tau) - x_i(\lambda_\pi(\tau))]\alpha_j(\tau) + [x_j(\tau) - x_j(\lambda_\pi(\tau))]\alpha_i(\tau)\} d\tau \\
&+ \sum_{r=1}^{m} \int_a^b \{[x_i(\tau) - x_i(\lambda_\pi(\tau))]\beta_{jr}(\tau) + [x_j(\tau) - x_j(\lambda_\pi(\tau))]\beta_{ir}(\tau)\} dz_r(\tau).
\end{aligned}$$

Next,

$$\begin{aligned}
&\left| \int_a^b [x_i(\tau) - x_i(\lambda_\pi(\tau))]\alpha_j(\tau) d\tau \right|^2 \\
&\leq \left(\int_a^b |(x_i(\tau) - x_i(\lambda_\pi(\tau))|^2 d\tau \right) \left(\int_a^b \alpha_j^2(\tau) d\tau \right)
\end{aligned}$$

by the Schwarz inequality. With probability 1 the second factor is finite and the first factor goes to zero as $|\pi| \longrightarrow 0$ by the Bounded Convergence Theorem, since $x_i(\cdot, \omega)$ is continuous and bounded on $[a, b]$. So

$$\lim_{|\pi| \to 0} \int_a^b \{[x_i(\tau) - x_i(\lambda_\pi(\tau))]\alpha_j(\tau) + [x_j(\tau) - x_j(\lambda_\pi(\tau))]\alpha_i(\tau)\} d\tau = 0$$

with probability 1. Let

$$W(\pi, r;\ \tau) = (x_i(\tau) - x_i(\lambda_\pi(\tau))\beta_{jr}(\tau).$$

To complete the proof it must be shown that $\int_a^b W(\pi, r;\ \tau)dz_r(\tau) \longrightarrow 0$ in probability as $|\pi| \longrightarrow 0$. By the sample continuity of the x_i-process, $W(\pi,\ r;\ \tau) \longrightarrow 0$ with probability 1 as $|\pi| \longrightarrow 0$. To obtain the desired result we appeal to the Dominated Convergence Theorem for stochastic integrals [18, p. 19]. For this, it suffices to exhibit a stochastic process $\{M(t),\ t \in R^+\}$ satisfying (C1)-(C3) such that

$$|W(\pi,\ r;\ t)| \leq |M(t)|, \quad a \leq t \leq b,$$

with probability 1.

Let

$$M_1(t) = \max\{x_i(u):\ 0 \leq u \leq t\}.$$

Since almost all sample functions $x_i(\cdot, \omega)$ are continuous on R^+, $M_1(\cdot, \omega)$ is continuous and monotone non-decreasing on R^+ with probability 1. Also $M_1(t)$ is a random variable for each $t \in R^+$ since the x_i-process is separable. Therefore, by a theorem of Doob [4, p. 60], $M_1(\cdot, \cdot)$ is a measurable function on $R^+ \times \Omega$. Furthermore, $M(t)$ is a function of $\{x_i(u),\ 0 \leq u \leq t\}$ and so is $\mathscr{F}(0, t)$-measurable. Let

$$M(t) = M_1(t)\beta_{ir}(t).$$

$\{M(t),\ t \in R^+\}$ satisfies (C1) and (C2) since the M_1- and β_{ir}-processes do. Moreover, it satisfies (C3) because if $T > 0$,

$$\begin{aligned} \int_0^T M^2(t, \omega)dt &= \int_0^T M_1^2(t, \omega)|\beta_{ir}(t, \omega)|^2 dt \\ &\leq M_1^2(T, \omega)\int_0^T \beta_{ir}^2(t, \omega)dt < \infty \end{aligned}$$

with probability 1 by (C3) for the β_{ir}-process. Finally, it is clear from the definition of $M(t, \omega)$ that if $t \in [a, b]$,

$$|W(\pi,\ r;\ t)| \leq |M(t)|$$

with probability 1. This completes the proof.

COROLLARY 4. 2. *Let the hypotheses of Theorem* 4. 1 *hold. Then there exists an increasing sequence* $\{n_1, n_2, \cdots\}$ *of positive integers such that*

$$\lim_{k\to\infty}\sum_{p=0}^{2^{n_k}-1} |x_i(a+2^{-n_k}(b-a)(p+1)) - x_i(a+2^{-n_k}(b-a)p)|^2$$
$$= \sum_{j=1}^{m}\int_a^b \beta_{ij}^2(s)ds$$

with probability 1.

This follows immediately from the second part of Theorem 4. 1 since any sequence converging in probability has an almost everywhere convergent subsequence.

Now let $\{Y(t),\ t\in R^+\}$ be the solution process of the stochastic differential equation (2. 1) or (2. 2). Set

$$\alpha_i(s,\omega) = m_i[s, Y(s,\omega)],\ \ \beta_{ij}(s,\omega) = \sigma_{ij}[s, Y(s,\omega)].$$

Then $\{\alpha_i(t),\ t\in R^+\}$, $\{\beta_{ij}(t),\ t\in R^+\}$ satisfy (C1)-(C3) so that by Corollary 4. 2,

$$\lim_{k\to\infty}\sum_{p=0}^{2^{n_k}-1} |y_i(a+2^{-n_k}(b-a)(p+1)) - y_i(a+2^{-n_k}(b-a)p)|^2$$
$$= \sum_{j=1}^{m}\int_a^b \sigma_{ij}^2(u, Y(u))du$$

with probability 1.

THEOREM 4. 3. *If for each $(t,x)\in R^+\times R^n$, $S(t,x)$ is different from the zero matrix, then the (vector) process Y, which is the solution of* (2. 1) *under conditions* (A1)-(A5), *is of unbounded variation on every interval $[a,b]\,(a<b)$ in R^+ with probability* 1. *If for each $(t,x)\in R^+\times R^n$, the row vector $(\sigma_{i1}(t,x),\ \cdots,\ \sigma_{im}(t,x))$ is different from the zero vector, then the y_i-process is of unbounded variation on every interval $[a,b]\,(a<b)$ in R^+ with probability* 1.

Proof. Recall that the Y-process is of unbounded variation on $[a,b]$ with probability 1 if

$$\sup_\pi \sum_{p=0}^{k-1} |Y(t_{p+1},\omega) - Y(t_p,\omega)| = \infty \ \text{ a.e. } [P]$$

where the supremum is taken over all partitions π of $[a,b]$ as in (4. 1). Let $0\le a<b<\infty$ be fixed but arbitrary. Let

$$P_i(\omega) = \int_a^b \sum_{j=1}^{m} \sigma_{ij}^2(u, Y(u,\omega))du,\ \ i=1,\ \cdots,\ n,$$
$$I(\omega) = \{i:\ P_i(\omega)>0\}.$$

Let $\{n_1, n_2, \cdots\}$ be as in Corollary 4. 2 and let Ω_0 be the set of all $\omega \in \Omega$ such that $Y(\cdot, \omega)$ is continuous on $[a, b]$ and

$$\lim_{k\to\infty} \sum_{p=0}^{r_k} |\Delta y(i, p, k)|^2 = P_i(\omega), \quad i = 1, \cdots, n,$$

where $r_k = 2^{n_k} - 1$ and

$$\Delta y(i, p, k) = y_i(a + 2^{-n_k}(b-a)(p+1)) - y_i(a + 2^{-n_k}(b-a)p).$$

Then $P(\Omega_0) = 1$. Let $\eta > 0$ be given. Then if $\omega \in \Omega_0$ and $i \in I(\omega)$ there exists a $\delta = \delta(\eta, \omega, i) > 0$ such that

$$|y_i(t, \omega) - y_i(s, \omega)| < \eta P_i(\omega) \tag{4. 2}$$

if $s,\ t \in [a, b]$, $|t - s| < \delta$, because $y_i(\cdot, \omega)$ is uniformly continuous on $[a, b]$.

Consequently there is an integer K_0 such that for $k \geq K_0$, $2^{-n_k} < \delta$ and so, for $p = 1, \cdots, 2^{n_k}$,

$$|\Delta y(i, p, k)|^2 \leq \eta P_i(\omega) |\Delta y(i, p, k)| \tag{4. 3}$$

by (4. 2). Also there is a K_1 such that for $k \geq K_1$,

$$\sum_{p=0}^{r_k} |\Delta y(i, p, k)|^2 > P_i(\omega)/2.$$

Let $K = K(\eta, \omega, i)$ be the larger of K_0, K_1. Then for each $k \geq K$,

$$\begin{aligned}\sum_{p=0}^{r_k} |\Delta y(i, p, k)| &\geq \{\sum_{p=0}^{r_k} |\Delta y(i, p, k)|^2\}/\eta P_i(\omega) \text{ by (4. 3)} \\ &\geq (P_i(\omega)/2)(\eta P_i(\omega))^{-1} = 1/2\eta.\end{aligned}$$

Since $\eta > 0$ is arbitrary the following conclusions are valid: first, if $I(\omega)$ is non-empty for each ω in a subset of Ω_0 of probability 1, then

$$\sup_\pi \sum_{p=0}^{k-1} |Y(t_{p+1}, \omega) - Y(t_p, \omega)| = \infty$$

with probability 1; second, if $i \in I(\omega)$ for each ω in a subset of Ω_0 of probability 1, then

$$\sup_\pi \sum_{p=0}^{k-1} |y_i(t_{p+1}, \omega) - y_i(t_p, \omega)| = \infty$$

with probability 1. Theorem 4. 3 is thus proved.

An immediate consequence of this theorem is

COROLLARY 4. 4. *Consider the second order Itô process described by equations* (3. 1). *Suppose* $\sigma(t,\xi,\eta)\neq 0$ *for each* t,ξ,η. *Then the sample functions* $y'(\cdot,\omega)$ *of the derived process are of unbounded variation in every interval* $[a,b]$ *with* $0\leq a<b<\infty$ *with probability* 1.

The results stated in Theorems 4. 1 and 4. 3 contain the results of Wong and Zakai [21]. See also Berman [1], Fisk [9]. The idea of the proof of Theorem 4. 1 is due to Wang [20].

THEOREM 4. 5. *Let the hypotheses of Theorem* 4. 1 *hold. Let* g *be a real-valued function, continuous on* R^1, *such that* $g(0)=0$ *and* $g''(0)$ *exists. Then the limit in probability of*

$$\sum_{p=0}^{k-1} g(x_i(t_{p+1})-x_i(t_p))$$

as $|\pi|\longrightarrow 0$ *exists and equals*

$$g'(0)\,(x_i(b)-x_i(a))+2^{-1}g''(0)\sum_{k=1}^{m}\int_a^b \beta_{ik}{}^2(s)ds.$$

If we set $i=j$ in Theorem 4. 1, then the resulting statement is the special case of the above theorem obtained by setting $g(x)=x^2$. On the other hand, Theorem 4. 5 is actually a consequence of Theorem 4. 1. The proof can be constructed along the lines of the proof of the corresponding result in Wang [20]. The details are omitted.

One of the drawbacks of the Itô stochastic integral is that it does not have the same formal properties as ordinary integrals. For example, in the formula for integration by parts, extra terms appear (see [4, p. 443], [14, p. 41]). For a particular example, we note that if $\{z(t),\ t\in R^+\}$ is a normalized Brownian motion process with $z(0)=0$, then

$$2\int_0^t z(s)dz(s)=z(t)^2-2t.$$

Stratonovich [19], Fisk [8], and Gray and Caughey [12] have (apparently independently) introduced a "symmetrized" stochastic integral which has the formal properties of ordinary integrals. Theorem 4. 1 has an intimate connection with the theory of symmetrized stochastic integrals. Before pointing out this connection we shall recall briefly the definition and some properties of symmetrized integrals. Out discussion follows Fisk [8].

Let $\{\mathscr{F}_t,\ t \in R^+\}$ be an increasing family of sub σ-fields of $\mathscr{F}$ and let $\{M(t),\ t \in R^+\}$, $\{N(t),\ t \in R^+\}$ be stochastic processes such that $M(t)$ and $N(t)$ are $\mathscr{F}_t$-measurable for each $t \in R^+$. For definiteness we shall suppose that $\{M(t),\ \mathscr{F}_t,\ t \in R^+\}$ and $\{N(t),\ \mathscr{F}_t,\ t \in R^+\}$ are sample continuous martingales. (More generally, the M- and N-processes can be quasi-martingales [7], [8]. For a different generalization, see Meyer [16, pp. 72-162].) If π is a partition of $[a, b] \subset R^+$ as in (4. 1), then the stochastic integral

$$(I)\int_a^b N(t)dM(t) = \lim_{|\pi|\to 0} \sum_{p=0}^{k-1} N(t_p)\,(M(t_{p+1}) - M(t_p))$$

will be called an *Itô-Doob integral* and it exists as a limit in probability. The stochastic integral

$$(S)\int_a^b N(t)dM(t) = \lim_{|\pi|\to 0} \sum_{p=0}^{k-1} 2^{-1}(N(t_p) + N(t_{p+1}))\,(M(t_{p+1}) - M(t_p))$$

will be called a *symmetrized integral* and it exists as a limit in probability.

The integration by parts formula for symmetrized integrals is valid:

$$(S)\int_a^b N(t)dM(t) + (S)\int_a^b M(t)dN(t) = N(b)M(b) - N(a)M(a).$$

If f and its first two derivatives are bounded continuous functions on R^1, then

$$f(M(b)) - f(M(a)) = (S)\int_a^b f'(M(t))dM(t).$$

These formulas are not valid for Itô-Doob integrals. If the M-process is a Brownian motion process, then $\left\{(I)\int_0^t N(s)dM(s),\ \mathscr{F}_t,\ t \in R^+\right\}$ is a martingale, whereas $\left\{(S)\int_0^t N(s)dM(s),\ \mathscr{F}_t,\ t \in R^+\right\}$ is not a martingale in general.

It seems that for some purposes it is preferable to use Itô-Doob integrals, whereas symmetrized integrals are better suited for other purposes. For a discussion of the physical interpretation of the difference between Itô-Doob and symmetrized stochastic integrals, see [12].

The connection between the two types of stochastic integrals is given by the formula

$$(S)\int_a^b N(t)dM(t) = (I)\int_a^b N(t)dM(t) + \int_a^b dN(t)dM(t)$$

where

$$\int_a^b dN(t)dM(t) = \lim_{|\pi|\to 0} \sum_{p=0}^{k-1} (N(t_{p+1}) - N(t_p))(M(t_{p+1}) - M(t_p)).$$

Theorem 4. 1 enables us to evaluate this expression explicitly if $\{N(t),\ t\in R^+\}$ and $\{M(t),\ t\in R^+\}$ are components of solution processes of stochastic differential equations of the type (1. 1). In particular, Theorem 4. 1 enables us to evaluate $(S)\int_a^b x_i(t)dz(t)$ (where the x_i- and z-processes are as in Theorem 4. 1) by evaluating $(I)\int_a^b x_i(t)dz(t)$.

5. Comparison of second order Itô process trajectories.

For $i=1,\ 2$ let $\{y_i(t),\ t\in R^+\}$ be a second order Itô process with drift and diffusion coefficients m_i and σ, and with the same Brownian disturbance z. In other words, for $i=1,\ 2$,

$$y_i(t) = y_i(0) + \int_0^t y_i'(s)ds$$

$$y_i'(t) = y_i'(0) + \int_0^t m_i[s, Y_i(s)]ds + \int_0^t \sigma[s, Y_i(s)]dz(s),$$

and (B1)-(B3) hold. We shall prove a theorem (Theorem 5. 1) which states, roughly, that if the initial conditions are the same, i.e. $y_1(0)=y_2(0)$, $y_1'(0)=y_2'(0)$, and if $m_1<m_2$, then $y_1(t)<y_2(t)$ and $y_1'(t)<y_2'(t)$ for $0<t<h$ with probability 1, where h is a positive random variable; if in addition $\sigma(t,\xi,\eta)\equiv\sigma(t,\eta)$ is independent of ξ, then $y_1(t)<y_2(t)$ and $y_1'(t)\leq y_2'(t)$ for $0<t<\infty$ with probability 1. Since $y_i(\cdot,\omega)$ may be interpreted as representing the position of a diffusing particle as a function of time, Theorem 5. 1 implies that increasing the drift coefficient (m) has the effect of giving the particle a "push" in the positive direction. So this theorem "justifies" the use of the term "drift coefficient" for m.

Recall that τ is called a *stopping time for the z-process* if $\tau\colon \Omega\longrightarrow[0,\infty]$ and for each $s\in R^+$ the event $\{\omega\colon \tau(\omega)>s\}$ $(=[\tau>s])$ is independent of the increments $\{z(t)-z(r),\ t\geq r\geq s\}$.

THEOREM 5. 1. *Let $\{y_i(t),\ t\in R^+\}$ be two second order Itô processes as above. Suppose that m_1, m_2, σ are continuous and that for each positive number c there are constants $\alpha>1/2$ and $C>0$ such that if $t\in R^+$, x_1, $x_2\in R^2$, and $|x_1|$, $|x_2|\leq c$, then*

$$|\sigma(t,x_1)-\sigma(t,x_2)|\leq C|x_1-x_2|^\alpha. \tag{5. 1}$$

Suppose also that $m_1(t,x) < m_2(t,x)$ *for all* $(t,x) \in R^+ \times R^2$.

(a) *Let* τ *be a stopping time for the z-process such that* $Y_1(\tau(\omega), \omega) = Y_2(\tau(\omega), \omega)$ *for almost all (a. a.)* $\omega \in [\tau < \infty]$. *Then there exists a positive random variable* h_0 *such that*

$$y_2'(t,\omega) > y_1'(t,\omega) \text{ and } y_2(t,\omega) > y_1(t,\omega)$$

for $\tau(\omega) < t < \tau(\omega) + h_0(\omega)$ *for a. a.* $\omega \in [\tau < \infty]$.

(b) *Suppose that* $y_1(0) \leq y_2(0)$, $y_1'(0) \leq y_2'(0)$ *with probability* 1 *and that* $\sigma(t,\xi,\eta) \equiv \sigma(t,\eta)$ *is independent of* ξ. *Then for a. a.* $\omega \in \Omega$ *there is a number* $h(\omega) > 0$ *such that* $y_2'(t,\omega) > y_1'(t,\omega)$ *for* $0 \leq t < h(\omega)$; *moreover,*

$$y_2'(t,\omega) \geq y_1'(t,\omega) \text{ and } y_2(t,\omega) > y_1(t,\omega)$$

for all $t > 0$ *with probability* 1.

This theorem will follow from the general result presented below and from the existence and uniqueness result quoted in section 3. Note that if $\tau \equiv 0$, then the condition $Y_1(\tau) = Y_2(\tau)$ in part (a) means that the two second order Itô processes have the same initial conditions.

For $i = 1, 2$, let $\{Y^i(t),\ t \in R^+\}$ be the solution process of the stochastic differential equation

$$Y^i(t) = Y^i(0) + \int_0^t M^i[s, Y^i(s)]ds + \int_0^t S[s, Y^i(s)]dZ(s).$$

(A1)-(A5) are assumed to hold. Note that S and the Brownian motion Z are the same for the Y^1- and Y^2-processes. For the rest of this section the j-th [or jk-th] component of a vector [or matrix] quantity will be denoted by a subscript j [or jk].

THEOREM 5. 2. *Let* Y^1, Y^2 *be as above. Suppose that* M^1, M^2, S *are (jointly) continuous on* $R^+ \times R^n$. *Suppose that for each* $c > 0$, $T > 0$, *there are constants* $\alpha = \alpha(c,T) > 1/2$, $C = C(c,T) > 0$ *such that*

$$|S_{jp}(t,x) - S_{jp}(t,y)| \leq C|x-y|^\alpha \tag{5. 2}$$

whenever $t \in [0,T]$, $|x| \leq c$, $|y| \leq c$, $p = 1, \cdots, m$. *Here* j *is a fixed integer between* 1 *and* n. *Suppose that* $M_j^1(t,x) < M_j^2(t,x)$ *for all* $(t,x) \in R^+ \times R^n$. *Let* τ *be a stopping time for the Z-process.*

(a) *If* $Y^1(\tau(\omega),\omega) = Y^2(\tau(\omega),\omega)$ *for a. a.* $\omega \in [\tau < \infty]$, *then there is a positive random variable* h_0 *such that* $Y_j^1(t,\omega) < Y_j^2(t,\omega)$ *for* $\tau(\omega) < t < \tau(\omega) + h_0(\omega)$ *and a. a.* $\omega \in [\tau < \infty]$.

(b) *Suppose that* (5. 2) *is replaced by the more restrictive inequality*

$$|S_{jp}(t, x) - S_{jp}(t, y)| \leq C|x_j - y_j|^{\alpha} \tag{5. 3}$$

for t, x, y, p, j *as above in* (5. 2). *If* $Y_j^1(\tau(\omega), \omega) = Y_j^2(\tau(\omega), \omega)$ *for a. a.* $\omega \in [\tau < \infty]$, *then there is a positive number* $h(\omega)$ *such that* $Y_j^1(t, \omega) < Y_j^2(t, \omega)$ *for* $\tau(\omega) < t < \tau(\omega) + h(\omega)$; *moreover,* $Y_j^1(t, \omega) \leq Y_j^2(t, \omega)$ *for* $\tau(\omega) < t < \infty$. *These statements hold for a. a.* $\omega \in [\tau < \infty]$.

Let us first prove Theorem 5. 1 assuming the validity of Theorem 5. 2. For $t \in R^+$ and $x = (\xi, \eta) \in R^2$ set $j = 2$,

$$Y^i(t) = \begin{pmatrix} y_i(t) \\ y_i'(t) \end{pmatrix}, \quad S(t, x) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(t, x) \end{pmatrix}, \quad M^i(t, x) = \begin{pmatrix} \eta \\ m_i(t, x) \end{pmatrix}, \quad Z(t) = \begin{pmatrix} z_0(t) \\ z(t) \end{pmatrix}$$

where the z_0-process is a (dummy) normalized Brownian motion process independent of the z-process and of $Y^1(0)$, $Y^2(0)$ as in section 3. If the hypotheses concering m_i, σ in Theorem 5. 1 hold, then so do the corresponding hypotheses concerning M^i, S in Theorem 5. 2 hold. (If $\sigma(t, \xi, \eta) \equiv \sigma(t, \eta)$ is in Theorem 5. 1 (b), then (5. 1) implies (5. 3).)

Let the hypotheses of Theorem 5. 1 (a) hold. Then $y_2'(t, \omega) > y_1'(t, \omega)$ for *a. a.* $\omega \in [\tau < \infty]$ and $\tau(\omega) < t < h_0(\omega)$ by Theorem 5. 2 (a). Integration yields $y_2(t, \omega) > y_1(t, \omega)$ for $\tau(\omega) < t < h_0(\omega)$ and *a. a.* $\omega \in [\tau < \infty]$, and so Theorem 5. 1 (a) follows.

Now let the hypotheses of Theorem 5. 1 (b) hold. Let $\tau(\omega) = \inf\{t \geq 0: y_2'(t, \omega) \leq y_1'(t, \omega)\}$. Then τ is a stopping time for the z-process (and hence for the Z-process), and by the continuity of the sample paths and the fact that $y_1'(0) \leq y_2'(0)$, we have $y_2'(\tau(\omega), \omega) = y_1'(\tau(\omega), \omega)$ for *a. a.* $\omega \in [\tau < \infty]$. By Theorem 5. 2 (b), there is a positive constant $h(\omega)$ such that $y_2'(t, \omega) > y_1'(t, \omega)$ for $\tau(\omega) < t < \tau(\omega) + h(\omega)$; also $y_2'(t, \omega) \geq y_1'(t, \omega)$ for $\tau(\omega) \leq t < \infty$. These statements are valid for *a. a.* $\omega \in [\tau < \infty]$. Integrating with respect to t yields the conclusion of Theorem 5. 1 (b) for *a. a.* $\omega \in [\tau < \infty]$. But for any $\omega \notin [\tau < \infty]$ for which $y_i'(\cdot, \omega)$ is continuous, $i = 1, 2$, we have $y_2'(t, \omega) > y_1'(t, \omega)$ for all $t \in R^+$ by definition of τ and also $y_2(t, \omega) > y_1(t, \omega)$ for all $t \in R^+$ by integration. So Theorem 5. 1 (b) follows.

The proof of Theorem 5. 2 is long and complicated. The theorem is a multidimensional extension of Skorokhod's result, which corresponds to the case $n = m = 1$ (see [18, Chapter 5] and [17]). The basic ideas of the proof are the same as in Skorokhod's work, but there are several important differences. A detailed proof is given here for the sake of completeness and reference.

Proof of theorem 5. 2. The proofs of parts (a) and (b) are the same except for slight changes, mainly notational. We shall prove part (b) and indicate the changes necessary for the proof of (a). The ω-variable will not be displayed in what follows.

Let $T>0$ be fixed but arbitrary. For notational convenience set

$$A^i(t) = M^i[t, Y^i(t)],\ \ B^i(t) = S[t, Y^i(t)],\ \ i = 1, 2.$$

Let the hypotheses of Theorem 5. 2 (b) hold. For $s \in R^+$ set $\Psi(s) = 1$ if and only if

(i) $s \geq \tau$,

(ii) $\inf\{A_j^2(u) - A_j^1(u):\ \tau \leq u \leq s\} > 2^{-1}(A_j^2(\tau) - A_j^1(\tau))$, and $\Psi(s)=0$ otherwise. $\{\Psi(s),\ s \in R^+\}$ is a stochastic process. For $k>0,\ c>0,\ s \in R^+$ set

$$\Psi_k^c(s) = I_{[0,c]}(\sup_{u \leq s}(|Y^1(u)| + |Y^2(u)|))I_{[0,T]}(s)I_{[\tau,\tau+k]}(s)\Psi(s).$$

Here I_G is the indicator function of the set G. The following lemma is the key result to be used in the proof.

LEMMA 5. 3. *Under the above hypotheses, if*

$$\lambda_{jp}(k) = \int \Psi_k^c(s)[B_{jp}{}^2(s) - B_{jp}{}^1(s)]dZ_p(s),$$

then

$$\lim_{k \to 0} k^{-1} \sum_{p=1}^{m} \lambda_{jp}(k) = 0$$

with probability 1.

Proof. Here $\int$ means $\int_0^\infty$, but it can be taken to be $\int_0^T$ since $\Psi_k^c(s)$ contains the factor $I_{[0,T]}(s)$. For $p = 1, \cdots, m$ we have

$$E\left\{\left|\int \Psi_k^c(s)[B_{jp}{}^2(s) - B_{jp}{}^1(s)]dZ_p(s)\right|^2\right\}$$

$$= E\left\{\int \Psi_k^c(s)|B_{jp}{}^2(s) - B_{jp}{}^1(s)|^2 ds\right\} \text{ by the stochastic integral isometry}$$

$$\leq C^2 E\left\{\int \Psi_k^c(s)|Y_j^2(s) - Y_j^1(s)|^{2\alpha} ds\right\} \text{ by (5. 3)}$$

$$\leq C^2E\Big\{\Big(\int\Psi_k^c(s)ds\Big)^{1-\alpha}\Big(\int\Psi_k^c(s)|Y_j^2(s)-Y_j^1(s)|^2ds\Big)^{\alpha}\Big\} \text{ by Hölder's inequality}$$

$$\leq C^2\Big(\int E\Big\{\Psi_k^c(s)\Big\}ds\Big)^{1-\alpha}\Big(\int E\Big\{\Psi_k^c(s)\,|\,Y_j^2(s)-Y_j^1(s)|^2\Big\}ds\Big)^{\alpha} \text{ by Hölder's inequality}$$

$$\leq C^2k^{1-\alpha}\Big(\int E\,\{\Psi_k^c(s)|Y_j^2(s)-Y_j^1(s)|^2\}ds\Big)^{\alpha} \tag{5. 4}$$

by the definition of $\Psi_k^c(s)$. In the above computation, the fact that $(\Psi_k^c(s))^2=\Psi_k^c(s)$ was used several times. The first time Hölder's inequality was used, it was used in the form $\|fg\|_1\leq\|f\|_a\|g\|_b$ with $a=(1-\alpha)^{-1}$, $b=\alpha^{-1}$, $f=\Psi_k^c$, and $g=\Psi_k^c|Y_j^2-Y_j^1|^{2\alpha}$. The second time it was used in the form $E[fg]\leq E^{1/a}[f^a]E^{1/b}[g^b]$ with $a=(1-\alpha)^{-1}$, $b=\alpha^{-1}$, $f=\Big(\int\Psi_k^c(t)\,dt)^{1-\alpha}$, and $g=\Big(\int\Psi_k^c(t)|Y_j^2(t)-Y_j^1(t)|^{2\alpha}dt\Big)^{\alpha}$; and Fubini's Theorem was used several times. (In proving part (a), (5. 2) should be used instead of (5. 3); then the subscripts j should be erased from the right hand side of (5. 4).)

Since $(\sum_{p=1}^m a_p)^2\leq m\sum_{p=1}^m a_p^2$, it follows from (5. 4) that

$$E\Big\{\Big|\sum_{p=1}^m\int\Psi_k^c(s)\,(B_{jp}^2(s)-B_{jp}^1(s))\,dZ_p(s)\Big|^2\Big\}$$
$$\leq mC^2k^{1-\alpha}\Big(\int E\Big\{\Psi_k^c(s)\,|\,Y_j^2(s)-Y_j^1(s)|^2\Big\}ds\Big)^{\alpha}. \tag{5. 5}$$

Observe that $\Psi_k^c(s)=1$ if and only if

(i′) $\sup\{|Y^1(t)|+|Y^2(t)|:\ 0\leq t\leq s\}\leq c$,

(ii′) $s\in[0,T]\cap[\tau,\tau+k]$,

(iii′) $\Psi(s)=1$.

Let $u\in[0,s]$. Then clearly (i′) holds for u. (ii′) holds for u if and only if $u\geq\tau$, in which case (iii′) also holds for u. Therefore, if $u\leq s$,

$$\Psi_k^c(s)=1 \text{ implies } \Psi_k^c(u)=I_{[\tau,\,s]}(u). \tag{5. 6}$$

For *a. a.* $\omega\in[\tau<\infty]$, $Y_j^1(\tau)=Y_j^2(\tau)$, and so for *a. a.* such ω, we have

$$Y_j^2(s)-Y_j^1(s)=\int_\tau^s(A_j^2(u)-A_j^1(u))du+\sum_{p=1}^m\int_\tau^s(B_{jp}^2(u)-B_{jp}^1(u))dZ_p(u)$$

for $s\geq\tau(\omega)$. (In proving part (a), this equation should be replaced by

$$Y^2(s)-Y^1(s)=\int_\tau^s(A^2(u)-A^1(u))du+\int_\tau^s(B^2(u)-B^1(u))dZ(u).\ )$$

Hence, using (5. 6),

$$\Psi_k^c(s)(Y_j^2(s) - Y_j^1(s)) = \Psi_k^c(s)\int_\tau^s \Psi_k^c(u)(A_j^2(u) - A_j^1(u))\,du$$

$$+ \sum_{p=1}^{m} \Psi_k^c(s)\int_\tau^s \Psi_k^c(u)(B_{jp}^2(u) - B_{jp}^1(u))dZ_p(u) \tag{5. 7}$$

for *a. a.* $\omega \in [\tau < \infty]$ for which $s \geq \tau(\omega)$. But if $s < \tau(\omega)$ or if $\omega \in [\tau = \infty]$ then both sides are zero, so (5. 7) holds with probability 1.

Since a continuous function on a compact set is bounded, there is a constant H such that

$$I_{[0,c]}(|x|)\,(|M^1(s,x)| + |M^2(s,x)|)I_{[0,T]}(s) \leq H$$

for all $x \in R^n$, $s \in R^+$.

Let $\lambda_{jp}(k)$ be as in the statement of the lemma and let $L(k) = \sum_{p=1}^{m} \lambda_{jp}(k)$. Note that it is required to show that $\lim_{k\to 0} k^{-1}L(k) = 0$ with probability 1. We have

$$E\{|L(k)|^2\} \leq mC^2k^{1-\alpha}\left[\int E\{\Psi_k^c(s)|Y_j^2(s) - Y_j^1(s)|^2\}ds\right]^\alpha \quad \text{by (5. 5)}$$

$$\leq 2^\alpha mC^2k^{1-\alpha}(J_1 + J_2)^\alpha \tag{5. 8}$$

where

$$J_1 = \int E\left\{\Psi_k^c(s)\left|\int_\tau^s \Psi_k^c(u)(A_j^2(u) - A_j^1(u))du\right|^2\right\}ds,$$

$$J_2 = \int E\left\{\Psi_k^c(s)\left|\int_\tau^s \Psi_k^c(u)\sum_{p=1}^{m}(B_{jp}^2(u) - B_{jp}^1(u))dZ_p(u)\right|^2\right\}ds$$

by (5. 7) and the trivial inequality $(a+b)^2 \leq 2(a^2+b^2)$. Hence

$$E\{|L(k)|^2\} \leq 2^{2\alpha}mC^2k^{1-\alpha}(J_1^\alpha + J_2^\alpha).$$

But

$$\Psi_k^c(u)|A_j^2(u) - A_j^1(u)| \leq H\Psi_k^c(u)$$

by the definition of the constant H. Therefore

$$J_1 \leq \int E\{\Psi_k^c(s)(Hk)^2\}ds \leq H^2k^3. \tag{5. 9}$$

Next, let

$$L_s(k) = \int_\tau^s \Psi_k^c(u)\sum_{p=1}^{m}(B_{jp}^2(u) - B_{jp}^1(u))dZ_p(u)$$

$$= \int_0^s I_{[\tau,T]}(u)\Psi_k^c(u)\sum_{p=1}^m (B_{jp}^2(u) - B_{jp}^1(u))dZ_p(u).$$

Then $\{L_s(k),\ s \in R^+\}$, being a sum of m independent martingales, is a martingale (see [4, p. 445], [18, p. 21]). Consequently

$$E\{|L(k)|^2\} \leq c_1 k^{1+2\alpha} + c_2 k^{1-\alpha}\left[\int E\{\Psi_k^c(s)|L_s(k)|^2\}ds\right]^\alpha \tag{5. 10}$$

where $c_1 = 2^{2\alpha}mC^2H^{2\alpha}$, $c_2 = 2^{2\alpha}mC^2$ (cf. (5. 8), (5. 9)). Note also that

$$\int \Psi_k^c(s)|L_s(k)|^2 ds \leq k \sup\{|L_s(k)|^2 :\ 0 \leq s \leq T\}.$$

Recall that if $\{X(t),\ t \in [0,T]\}$ is a square integrable martingale, then

$$E\{\sup|X(s)|^2 :\ 0 \leq s \leq T\} \leq 4E\{|X(T)|^2\} \tag{5. 11}$$

(see [18, p. 9], [4, p. 317]). Since $\{L_s(k),\ 0 \leq s \leq T\}$ is a square integrable martingale, it follows from (5. 10), (5. 11) that

$$E\{|L(k)|^2\} \leq c_1 k^{1+2\alpha} + c_3 k[E\{|L_T(k)|^2\}]^\alpha \tag{5. 12}$$

where $c_3 = 2^{2\alpha}c_1$. Set $\nu(k) = E\{|L(k)|^2\}$. Then, by the stochastic integral isometry and the independence of the Z_p-processes,

$$E\{|L_T(k)|^2\} = E\left\{\int I_{[\tau,T]}(u)\,\Psi_k^c(u)\left|\sum_{p=1}^m (B_{jp}^2(u) - B_{jp}^1(u)\right|^2 du\right\}$$

$$\leq E\left\{\int \Psi_k^c(u)\left|\sum_{p=1}^m (B_{jp}^2(u) - B_{jp}^1(u))\right|^2 du\right\} = E\{|L(k)|^2\}.$$

It follows from this and (5. 12) that

$$\nu(k) \leq c_1 k^{1+2\alpha} + c_3 k[\nu(k)]^\alpha. \tag{5. 13}$$

Let $\mu(k) = k^{-1-2\alpha}\nu(k)$. We claim that there are positive constants D, δ such that

$$\mu(k) \leq D \quad \text{if} \quad 0 < k \leq \delta. \tag{5. 14}$$

To prove this, suppose it is not true. Then there is a sequence $\{k_i\}$ of positive numbers such that $k_i \longrightarrow 0$ and $\mu(k_i) \longrightarrow \infty$ as $i \longrightarrow \infty$. Let $1/2 < \beta < \alpha \leq 1$ and let

$$D_0 = \{(t,x) :\ t \in [0,T],\ x \in R^n,\ |x| \leq c\}.$$

By hypothesis, $|S_{jp}|$ is bounded on D_0, say by C_1, and if (t,x), $(t,y) \in D_0$, $p = 1, \cdots, m$, we have, by (5. 3),

$$|S_{jp}(t,x)-S_{jp}(t,y)|\leq C|x_j-y_j|^{\alpha}\leq C|x_j-y_j|^{\beta}\quad\text{if}\quad |x_j-y_j|\leq 1,$$
$$|S_{jp}(t,x)-S_{jp}(t,y)|\leq 2C_1\leq 2C_1|x_j-y_j|^{\beta}\quad\text{if}\quad |x_j-y_j|>1.$$

Hence

$$|S_{jp}(t,x)-S_{jp}(t,y)|\leq C_0|x_j-y_j|^{\beta}$$

where $C_0=\max(C,2C_1)$. The point is that without loss of generality it may and will be assumed that $\alpha<1$. (In proving part (a), the subscript j should be erased in each x_j, y_j in the above equations.) Dividing both sides of (5. 13) by $[\mu(k)]^{\alpha}k^{1+2\alpha}$ and letting $k=k_i$ we obtain

$$[\mu(k_i)]^{1-\alpha}\leq c_1[\mu(k_i)]^{-\alpha}+c_3k_i^{\alpha(2\alpha-1)}.$$

Since $\alpha<1$ and $\mu(k_i)\longrightarrow\infty$, the left side of the above inequality tends to ∞ as $i\longrightarrow\infty$. But since $\alpha>1/2$ and $\mu(k_i)\longrightarrow\infty$, the right side tends to zero as $i\longrightarrow\infty$. This is the desired contradiction, and so the claim (5. 14) is proved.

Let

$$G_p(s)=I_{[0,T]}(s)I_{[0,c]}(\sup_{u\leq s}(|Y^1(u)|+|Y^2(u)|))\Psi(s)(B_{jp}{}^2(u)-B_{jp}{}^1(u)),$$
$$W_p(t)=\int_0^t G_p(s)dZ_p(s).$$

Note that $W_p(t)=W_p(T)$ for $t\geq T$. $\{W_p(t),\ t\in R^+\}$ is a martingale. Set $\tau(k)=\tau+k$. Then $\{W_p(\tau(k)),\ k\in R^+\}$ is a martingale by [18, p. 9]. Since

$$\lambda_{jp}(k)=\int I_{[\tau,\tau+k]}(s)\,G_p(s)dZ_p(s)=W_p(\tau(k))-W_p(\tau(0)),$$

it follows that $\{\lambda_{jp}(k),\ k\in R^+\}$ is a martingale. Hence $\{L(k)=\sum_{p=1}^{m}\lambda_{jp}(k),\ k\in R^+\}$, being a sum of independent martingales, is itself a martingale. By (5. 11) and (5. 14), if $0<k_0\leq\delta$, then

$$E\{\sup|L(k)|^2\colon 0\leq k\leq k_0\}\leq 4E\{|L(k_0)|^2\}\leq 4Dk_0^{1+2\alpha}.\tag{5. 15}$$

If $2^{-q-1}\leq k$, then $k^{-1}\leq 2^{q+1}$, and so

$$\begin{aligned}&P\{\sup|k^{-1}L(k)|>q^{-1}\colon 2^{-q-1}\leq k\leq 2^{-q}\}\\&\leq P\{\sup|L(k)|>q^{-1}2^{-q-1}\colon 0\leq k\leq 2^{-q}\}\\&\leq q^2 2^{2q+2}E\{\sup|L(k)|^2\colon 0\leq k\leq 2^{-q}\}\ \text{by Chebyshev's inequality}\\&\leq q^2 2^{2q+2}4D(2^{-q})^{1+2\alpha}\qquad\text{by (5. 15)}\\&=16Dq^2 2^{-q(2\alpha-1)}.\end{aligned}$$

But $\alpha > 1/2$, and so $\sum_{q=1}^{\infty} q^2 2^{-q(2\alpha-1)}$ converges; hence by the Borel-Cantelli Lemma [4, p. 104], it follows that $\lim_{k\to 0} k^{-1}L(k) = 0$ with probability 1. This completes the proof of Lemma 5.3.

Proof of Theorem 5.2 (continued). Note that by the definition of Ψ_k^c we have

$$\begin{aligned}
&\Psi_k^c(\tau+k)\int \Psi_k^c(u)(A_j^2(u)-A_j^1(u))du \\
&\geq \Psi_k^c(\tau+k)\int \Psi_k^c(u)2^{-1}(A_j^2(\tau)-A_j^1(\tau))du \\
&= 2^{-1}k\Psi_k^c(\tau+k)(A_j^2(\tau)-A_j^1(\tau)). \qquad (5.16)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Psi_k^c(\tau+k)(Y_j^2(\tau+k)-Y_j^1(\tau+k)) \\
&= \Psi_k^c(\tau+k)\int \Psi_k^c(u)(A_j^2(u)-A_j^1(u)) \\
&+ \Psi_k^c(\tau+k)\int \Psi_k^c(u)\sum_{p=1}^{m}(B_{jp}^2(u)-B_{jp}^1(u))dZ_p(u) \text{ by (5.7)} \\
&\geq k\Psi_k^c(\tau+k)\{2^{-1}(A_j^2(\tau)-A_j^1(\tau))+k^{-1}L(k)\} \qquad (5.17)
\end{aligned}$$

by (5.16) and the definition of $L(k)$. For *a. a.* $\omega\in[\tau<\infty]$, $A_j^2(\tau)>A_j^1(\tau)$ by hypothesis and $k^{-1}L(k)\longrightarrow 0$ as $k\longrightarrow 0$ by Lemma 5.3. For such ω there is a positive number $h(\omega)$ such that

$$k^{-1}|L(k)| < 4^{-1}(A_j^2(\tau)-A_j^1(\tau))$$

for $0<k<h(\omega)$. Therefore the right side of (5.17) is strictly positive for $0<k<h(\omega)$ and $\Psi_k^c(\tau+k)=1$. But for *a. a.* $\omega\in[\tau<\infty]$ there are positive numbers c, k_0 (depending on ω) such that $\Psi_k^c(\tau+k)=1$ for $0<k<k_0$. This completes the proof of the first assertion in part (*b*).

Let

$$h_1(\omega)=\inf\{t>\tau:\ Y_j^2(t)\leq Y_j^1(t)\}$$

if $\{t>\tau:\ Y_j^2(t)\leq Y_j^1(t)\}$ is non-empty; otherwise set $h_1(\omega)=\infty$. Then $h_0=h_1-\tau$ is an *a. e.* positive extended real-valued random variable and $Y_j^1(t)<Y_j^2(t)$ for $\tau<t<h_1=h_0+\tau$ for *a. a.* $\omega\in[\tau<\infty]$. (In fact, h_1 is the first exit time after τ of a certain Markov process from a certain open set. See below.) Thus part (a) is proved.

We now complete the proof of part (b). Let $\tau_1(\omega) = \infty$ if $Y_j^1(t) \leq Y_j^2(t)$ for all $t \in [\tau, \infty]$ or if $\tau(\omega) = \infty$; otherwise let $\tau_1(\omega) = \inf\{t: Y_j^1(t) > Y_j^2(t)\}$. By the continuity of the paths, $Y_j^1(\tau_1) = Y_j^2(\tau_1)$ for $a.\ a.$ $\omega \in [\tau_1 < \infty]$. But τ_1 is the first entrance time after τ of the Markov process $\{(Y^1(t), Y^2(t)), t \in R^+\}$ into the open set $\{(x, y): x, y \in R^n, x_j > y_j\}$. Hence by [5, Chapter 4] τ_1 is a stopping time for the (Y^1, Y^2)-process and hence for the Z-process. By the first part of part (b) of the theorem (with τ_1 in place of τ), for $a.\ a.$ $\omega \in [\tau_1 < \infty]$, we have $Y_j^1(t) < Y_j^2(t)$ for $\tau_1 < t < h_2$, where $h_2(\omega) > 0$. Thus we have a contradiction unless $P[\tau_1 < \infty] = 0$. This completes the proof.

6. **Moments of second order Itô processes.**

For notational convenience let

$$m_0(t) = m[t, y(t), y'(t)],\ \sigma_0(t) = \sigma[t, y(t), y'(t)]$$

for a second order Itô process. Then equations (3. 1) take the form

$$y(t) = y(0) + \int_0^t y'(s)ds$$

$$y'(t) = y'(0) + \int_0^t m_0(s)ds + \int_0^t \sigma_0(s)dz(s).$$

(B1)-(B3) are assumed to hold. In this section the mean and variance of $y(t)$, $y'(t)$ are computed since these statistics yield some information concerning the y-process and the derived process for large t. The elementary properties of stochastic integrals will be used without specific mention (see [4, Chapter IX]). First,

$$E[y'(t)] = E[y'(0)] + \int_0^t E[m_0(s)]ds, \tag{6. 1}$$

$$E[y(t)] = E[y(0)] + \int_0^t E[y'(s)]ds. \tag{6. 2}$$

Next,

$$[y'(t) - y'(0)]^2 = \left[\int_0^t m_0(s)\,ds\right]^2 + \left[\int_0^t \sigma_0(s)\,dz(s)\right]^2 + 2\int_0^t m_0(u)\,du \int_0^t \sigma_0(s)dz(s),$$

so that

$$E\{[y'(t) - y'(0)]^2\} = E\left\{\left[\int_0^t m_0(s)\,ds\right]^2\right\} + \int_0^t E\{\sigma_0^2(s)\}ds$$

since

$$E\left\{\int_0^t m_0(u)\,du\int_0^t \sigma_0(s)ds\right\} = \int_0^t E\left\{\int_0^t \sigma_0(s)m_0(u)\,dz(s)\right\}du = 0$$

by Fubini's Theorem, since the expectation of a stochastic integral is zero. (For a discussion of Fubini's Theorem when stochastic integrals are involved, see Chapter 3, section 3 of [2].)

Let $\mathrm{Var}(X)$ denote the variance of a random variable X. Then

$$\begin{aligned}
&\mathrm{Var}[y'(t)-y'(0)] = E\{[y'(t)-y'(0)]^2\} - E^2[y'(t)-y'(0)]\\
&= E\left\{\left[\int_0^t m_0(s)\,ds\right]^2\right\} + \int_0^t E\{\sigma_0^2(s)\}\,ds - E^2\left[\int_0^t m_0(s)\,ds\right]\\
&= \mathrm{Var}\left[\int_0^t m_0(s)\,ds\right] + \int_0^t E\{\sigma_0^2(s)\}ds. \qquad (6.\ 3)
\end{aligned}$$

For the y-process,

$$\begin{aligned}
y(t)-y(0) &= \int_0^t\left[y'(0) + \int_0^s m_0(u)du + \int_0^s \sigma_0(u)dz(u)\right]ds\\
&= ty'(0) + \int_0^t\int_u^t m_0(u)dsdu + \int_0^t\int_u^t \sigma_0(u)dsdz(u)\\
&= ty'(0) + \int_0^t (t-u)\,m_0(u)du + \int_0^t (t-u)\sigma_0(u)dz(u).
\end{aligned}$$

One immediate conclusion from this is that

$$E[y(t)-y(0)] = E\left\{ty'(0) + \int_0^t (t-u)m_0(u)du\right\}. \qquad (6.\ 4)$$

This equality, which will be used in computing $\mathrm{Var}[y(t)-y(0)]$, also follows easily from (6. 1) and (6. 2).

Let

$$A = ty'(0) + \int_0^t (t-u)m_0(u)du,\quad B = \int_0^t (t-s)\sigma_0(s)dz(s).$$

Then

$$E\{[y(t)-y(0)]^2\} = E[A^2] + E[B^2] + 2E[AB].$$

But

$$E[B^2] = \int_0^t (t-s)^2 E\{\sigma_0^2(s)\}ds$$

and

$$E[AB] = tE\left\{\int_0^t y'(0)\,(t-s)\sigma_0(s)dz(s)\right\} + \int_0^t (t-u)\; E\left\{\int_0^t (t-s)m_0(u)\sigma_0(s)dz(s)\right\}du$$
$$= 0.$$

Consequently,

$$\begin{aligned}\mathrm{Var}\,[y(t)-y(0)] &= E\{[y(t)-y(0)]^2\} - E^2[y(t)-y(0)]\\ &= E\left\{\left[ty'(0)+\int_0^t (t-u)m_0(u)du\right]^2\right\} + \int_0^t (t-u)^2 E\{\sigma_0^2(u)\}du\\ &\quad - E^2\left[ty'(0)+\int_0^t (t-u)m_0(u)du\right]\end{aligned}$$

by the above calculation and (6. 4)

$$= \mathrm{Var}\left[ty'(0)+\int_0^t (t-u)m_0(u)du\right] + \int_0^t (t-u)^2 E\{\sigma_0^2(u)\}du. \tag{6. 5}$$

Also,

$$\mathrm{Var}\,[y'(t)] = \mathrm{Var}\,[y'(t)-y'(0)] + \mathrm{Var}\,[y'(0)], \tag{6. 6}$$

$$\mathrm{Var}\,[y(t)] = \mathrm{Var}\,[y(t)-y(0)] + \mathrm{Var}\,[y(0)], \tag{6. 7}$$

since $y'(t)-y'(0)$ and $y(t)-y(0)$, which are measurable relative to the σ-field generated by $\{z(v)-z(u),\ 0\leq u\leq v\leq t\}$, are independent of $y(0)$, $y'(0)$ by (B3).

In particular, suppose that σ is bounded away from zero, i.e. $\sigma(t,x)\geq\gamma$ for all $(t,x)\in R^+\times R^2$ and some $\gamma>0$. Then by (6. 3) and (6. 5) – (6. 7),

$$\mathrm{Var}\,[y'(t)]\geq \mathrm{Var}\,[y'(t)-y'(0)]\geq\int_0^t E\{\sigma_0^2(u)\}du\geq\gamma^2 t,$$

$$\mathrm{Var}\,[y(t)]\geq \mathrm{Var}\,[y(t)-y(0)]\geq\int_0^t (t-u)^2 E\{\sigma_0^2(u)\}du\geq\gamma^2 t^3/3.$$

Therefore $\mathrm{Var}\,[y'(t)]$ and $\mathrm{Var}\,[y(t)]$ tend to infinity as $t\longrightarrow\infty$. In particular, neither the y-process nor the y'-process can have a compact state space.

Even if σ is not bounded away from zero, there are still cases in which

$$\lim_{t\to\infty}\mathrm{Var}\,[y'(t)] = \lim_{t\to\infty}\mathrm{Var}\,[y(t)] = \infty.$$

For a simple example, let $m\equiv 0$ and $y'(0)\neq$ constant; then

$$\begin{aligned}\mathrm{Var}\,[y(t)]\geq \mathrm{Var}\,[y(t)-y(0)] &\geq \mathrm{Var}\left[ty'(0)+\int_0^t (t-u)m_0(u)du\right]\\ &= t^2\,\mathrm{Var}\,[y'(0)]\longrightarrow\infty \text{ as } t\longrightarrow\infty,\end{aligned}$$

no matter what σ is.

7. The case of non-random σ. In case $\sigma(t,x)\equiv\sigma(t)$, so that σ is a function of t alone, a more detailed analysis of the sample paths of a second order Itô process can be given. An example of such a process is the solution process for the equation of the Brownian oscillator. In this case $\sigma\equiv 1$ and $m[t,\xi,\eta]=-2\alpha\eta-\beta^2\xi$, where α, β are constants satisfying $0<\alpha<\beta$. For more information on this example see Edwards and Moyal [6] and, for a generalization of [6], see Goldstein [11].

PROPOSITION 7.1. *Let $\{y(t),\ t\in R^+\}$ be a second order Itô process with drift and diffusion coefficients m, σ such that* (B1) – (B3) *hold. Suppose that σ is a positive function of time alone, so that $\sigma(t,x)=\sigma(t)>0$ independent of $x\in R^2$, and assume that $\int_0^\infty\sigma^2(u)du=\infty$.*

(a) *If $m\equiv 0$ then $\{y'(t),\ t\in R^+\}$ is a Brownian motion process after a change of the time scale.*

(b) *Suppose there is a positive constant γ such that*

$$m[t,x]\geq\gamma \qquad [\text{or } m[t,x]\leq-\gamma]$$

for all sufficiently large t and all $x\in R^2$. If also there is a $\delta_0>0$ such that

$$2\log\log\int_0^t\sigma^2(u)du\leq(\gamma-\delta_0)^2t^2$$

for all sufficiently large t, then

$$\lim_{t\to\infty}y'(t)=\infty \qquad [\lim_{t\to\infty}y'(t)=-\infty],$$

$$\lim_{t\to\infty}y(t)=\infty \qquad [\lim_{t\to\infty}y(t)=-\infty]$$

with probability 1.

Proof. Let $\sigma\colon R^+\longrightarrow(0,\infty)$ be a Baire function which is bounded on compact subsets of R^+. Let $\mathscr{F}(s,t)$ be the σ-field generated by the increments $\{z(v)-z(u),\ s\leq u\leq v\leq t\}$. Let

$$x(t)=\int_0^t\sigma(u)dz(u).$$

Then, for $t\geq s\geq 0$,

$$x(t)-x(s)=\int_s^t\sigma(u)dz(u)$$

is $\mathscr{F}(s,t)$ measurable. It follows that the x-process has independent increments. Moreover, by standard properties of stochastic integrals, $\{x(t),\ t\in R^+\}$ is a martingale with continuous paths. Hence, by a well-known result [4, p. 420], the x-process becomes a Brownian motion process after a scale change. More precisely, let

$$\xi(t) = \mathrm{Var}\,[x(t)] = E\{[x(t)]^2\}.$$

Then $\xi(t) = \int_0^t \sigma^2(u)du$, and ξ is continuous and strictly increasing. If $\eta=\xi^{-1}$ is the inverse function, then $\{b(t) = x(\eta(t)),\ t \in R^+\}$ is a Brownian motion process with unit variance parameter. This proves part (a).

It is easy to translate the sample function properties of a Brownian motion process to those of the x-process. For example, in case $\int_0^\infty \sigma^2(u)du=\infty$, then it follows from the Law of the Iterated Logarithm [15, p. 242] that

$$\overline{\lim_{t\to\infty}}\, x(t) = \infty, \quad \underline{\lim_{t\to\infty}}\, x(t) = -\infty$$

with probability 1. Also $x(t) - x(s) = b(\xi(t)) - b(\xi(s))$ is normally distributed with mean zero and variance $|\xi(t) - \xi(s)| = \left|\int_s^t \sigma^2(u)du\right|$. Hence $x(t) - x(s)$ has all of R^1 as its state space; so does $x(t)$ since $x(0) = 0$.

Let the hypotheses of part (b) hold. Then

$$y'(t) = y'(0) + x_0(t) + x(t) \tag{7. 1}$$

where $x_0(t) = \int_0^t m[s, y(s),\ y'(s)]ds$ and the x-process is as above. The x-process is a Brownian motion after a scale change, so that its same function behavior is well-known. By the hypothesis on m, given $\delta > 0$, there is a $T_0 = T_0(\delta, \omega) > 0$ such that

$$x_0(t) \geq (\gamma - \delta)t \qquad [\text{or } x_0(t) \leq (\gamma - \delta)t]$$

for all $t \geq T_0$ with probability 1. The hypothesis

$$(2 \log \log \xi(t))^{1/2} \leq (\gamma - \delta_0)t$$

together with the Law of the Iterated Logarithm and the decomposition (7. 1) imply

$$\lim_{t\to\infty} y'(t) = \infty \qquad [\text{or } \lim_{t\to\infty} y'(t) = -\infty]$$

with probability 1. Integration yields

$$\lim_{t\to\infty} y(t) = \infty \qquad [\text{or } \lim_{t\to\infty} y(t) = -\infty]$$

with probability 1.

Hence, "consistent drift" implies that the velocity ($y'-$) and the displacement ($y-$) processes both tend to $\pm\infty$ as $t \longrightarrow \infty$, the $\pm$ sign being the same as the sign of m for large t.

It seems likely that part (b) is true even if σ is non-random, but we have no proof of this conjecture.

8. The stationary case: Associated semi-groups and martingales. In this section the stationary second order Itô processes described by

$$\begin{aligned} y(t) &= y(0) + \int_0^t y'(s)ds \\ y'(t) &= y'(0) + \int_0^t m[y(s),\ y'(s)]ds + \int_0^t \sigma[y(s),\ y'(s)]dz(s) \end{aligned} \tag{8. 1}$$

will be considered. (B1) — (B3) are assumed to hold. Here "stationarity" means that the Markov process $\left\{Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix},\ t \in R^+\right\}$ has a stationary transition function, and this is the case whenever $m[t, \xi, \eta]$ and $\sigma[t, \xi, \eta]$ are both independent of t.

Let $\mathscr{C}$ be the space of all real-valued continuous functions on R^2 that vanish at infinity. (We could equally well treat the complex-valued case.) $\mathscr{C}$ is a Banach space under the supremum norm. For $f \in \mathscr{C}$ and $x \in R^2$ define

$$(\mathscr{T}_t f)(x) = E\{f[Y(t)] | Y(0) = x\};$$

stated differently,

$$(\mathscr{T}_t f)(x) = \int_{R^2} f(y) P(t, x, dy)$$

where $P(t, x, G) = P\{Y(t) \in G | Y(0) = x\}$ is the transition function of the Y-process. $\{\mathscr{T}_t,\ t \in R^+\}$ is a strongly continuous contraction semi-group of linear operators on $\mathscr{C}$ (see Dynkin [5, p. 349]). (In other words, the Y-process is a Feller process.)

Let $\mathscr{A}$ denote the infinitesimal generator of $\{\mathscr{T}_t,\ t \in R^+\}$; $f \in D(\mathscr{A})$, the domain of $\mathscr{A}$, and

$$\mathscr{A} f = \lim_{h\to 0} h^{-1}(\mathscr{T}_h f - f)$$

whenever this limit exists in the norm topology of $\mathscr{C}$.

Let $\mathscr{C}_c^2$ be the collection of all twice continuously differentiable functions which have compact support in R^2. Then $\mathscr{C}_c^2 \subset D(\mathscr{A})$, and for $f \in \mathscr{C}_c^2$ and $(\xi, \eta) \in R^2$,

$$(\mathscr{A}f)(\xi, \eta) = 2^{-1}\sigma^2(\xi, \eta)(\partial^2 f/\partial\eta^2) + m(\xi, \eta)(\partial f/\partial\eta) + \eta(\partial f/\partial\xi). \tag{8. 2}$$

For proofs see [5, p. 352].

We shall also consider the condition

(B4) m, σ are uniformly bounded functions on R^2.

Let (B1) – (B4) hold. Let C be the set of all continuous real-valued functions f on R^2 such that

$$\|f\| = \sup\{|f(\xi, \eta)|e^{-|\eta|}: (\xi, \eta) \in R^2\} < \infty.$$

For $t \in R^+$, $x \in R^2$, and $f \in C$ define

$$(T_t f)(x) = E\{f[Y(t)]|Y(0) = x\} = \int_{R^2} f(y)P(t, x, dy).$$

Then, as Borchers showed in [2], C is a Banach space under the norm $\|.\|$, and $\{T_t, t \in R^+\}$ is a (not necessarily strongly continuous) contraction semigroup of linear operators on C. Define the infinitesimal operator A by

$$Af = \lim_{h\to 0} h^{-1}(T_t f - f),$$

the limit being in the norm topology of C. The domain $D(A)$ of A is the set of all $f \in C$ for which the above limit exists.

If $f \in C$, then we shall write $f \in C_{(2)}$ if and only if the following three conditions hold:

(i) f has continuous second partials everywhere in R^2.

(ii) There is a constant $c \in [0, 1)$ (depending on f) such that

$$\sup\{|(D^\alpha f)(\xi, \eta)|e^{-c|\eta|}: (\xi, \eta) \in R^2\} < \infty,$$

where $D^\alpha f$ stands for any derivative of f of order two or less.

(iii) There is a compact set $S \subset R^2$ (depending on f) such that f has continuous third partials in $R^2\backslash S$ and

$$\sup\{|(D^\alpha f)(\xi, \eta)|e^{-c|\eta|}: (\xi, \eta) \in R^2\backslash S\} < \infty$$

for each third order derivative $D^\alpha f$ of f; here c is the constant of condition (ii).

Borchers [2] proved that $C_{(2)} \subset D(A)$ and

$$(Af)(\xi,\eta) = 2^{-1}\sigma^2(\xi,\eta)(\partial^2 f/\partial\eta^2) + m(\xi,\eta)(\partial f/\partial\eta) + \eta(\partial f/\partial\xi)$$

for $f \in C_{(2)}$, $(\xi,\eta) \in R^2$.

In fact, in the general case in which m, σ are time dependent and (B4) is replaced by

(B4′) m, σ are uniformly bounded on $R^+ \times R^2$,

then, using ideas similar to those in [2], it can be shown that for $f \in C_{(2)}$, $t \in R^+$, $(\xi,\eta) \in R^2$,

$$\lim_{h\to 0} h^{-1}(T(t,\ t+h)f - f)(\xi,\eta)$$
$$= 2^{-1}\sigma^2(t,\xi,\eta)(\partial^2 f/\partial\eta^2) + m(t,\xi,\eta)(\partial f/\partial\eta) + \eta(\partial f/\partial\xi)$$

where for $t \in [0, s]$,

$$(T(t,s)f)(\xi,\eta) = E\{f[Y(s)] \mid Y(t) = (\xi,\eta)\}$$
$$= \int_{R^2} f(u,v)\, P(t,(\xi,\eta);\ s,\ \mathrm{dudv})$$

and $P(\cdot,\ \cdot;\ \cdot,\ \cdot)$ is the transition function of the (non-stationary) Markov process $\{Y(t),\ t \in R^+\}$.

THEOREM 8. 1. *Consider a second order Itô process described by* (8. 1) *for which* (B1) – (B3) *hold. Let* $g \in D(\mathscr{A})$ *satisfy either*

$(\mathrm{I})_1$ $\mathscr{A}g = \lambda$

or

$(\mathrm{II})_1$ $\mathscr{A}g = \lambda g$

for some real λ. *Then the corresponding stochastic process*

$(\mathrm{I}')_1$ $\{g[Y(t)] - \lambda t,\ t \in R^+\}$

or

$(\mathrm{II}')_1$ $\{e^{-\lambda t}g[Y(t)],\ t \in R^+\}$

is a martingale. If "$=$" *is replaced by* "$\geq$" *in either* $(\mathrm{I})_1$ *or* $(\mathrm{II})_1$, *then the corresponding stochastic process is a submartingale if its random variables have finite expectations* (i.e. *belong to* $L^1(\Omega, \mathscr{F}, P)$).

THEOREM 8. 2. *Consider a second order Itô process described by* (8. 1) *for which* (B1) – (B4) *hold, and suppose that* $E\{e^{|y'(0)|}\} < \infty$. *Let* $g \in D(A)$ *satisfy either*

$(\mathrm{I})_2 \quad Ag = \lambda$

or

$(\mathrm{II})_2 \quad Ag = \lambda g$

for some real λ. *Then the corresponding stochastic process*

$(\mathrm{I}')_2 \quad \{g[Y(t)] - \lambda t,\ t \in R^+\}$

or

$(\mathrm{II}')_2 \quad \{e^{-\lambda t} g[Y(t)],\ t \in R^+\}$

is a martingale. If "$=$" *is replaced by* "$\geq$" *in either* $(\mathrm{I})_2$ *or* $(\mathrm{II})_2$, *then the corresponding stochastic process is a submartingale if its random variables have finite expectations.*

Theorem 8. 2 is a reformulation of a result of Doob [3, pp. 190-191] adapted to the case of a second order Itô process with a stationary transition function. Doob's proof is valid without essential change for the present case, using some estimates of [2]. The proof of Theorem 8. 2 is presented below for completeness. Replacing A, T_t in the proof by $\mathscr{A}$, $\mathscr{T}_t$ yields a proof of Theorem 8. 1.

Proof. Let $b > 0$, $t \in [0, b]$, and $g \in D(A)$. Then

$$\begin{aligned}(T_{b-t}g)[Y(t)] &= E\{g[Y(b)] \mid Y(t)\} \quad \text{by stationarity}\\ &= E\{g[Y(b)] \mid Y(s),\ s \leq t\}\end{aligned}$$

by the Markov property. Thus $(T_{b-t}g)[Y(t)]$, for $0 \leq t \leq b$, is the conditional expectation of $g[Y(b)]$ with respect to an increasing family of σ-fields, and hence $\{(T_{b-t}g)[Y(t)],\ t \in [0, b]\}$ is a martingale if its random variables have finite expectations. If $(\mathrm{II})_2$ holds, so that $Ag = \lambda g$, then $T_t g = e^{\lambda t} g$ since each of $T_t g$ and $e^{\lambda t} g$ is the unique solution of $(d/dt) f(t) = \lambda f(t)$, $f(0) = g$. The reasoning is similar in case $(\mathrm{I})_2$ holds. Consequently, if $(\mathrm{II})_2$ holds,

$$(T_{b-t}g)[Y(t)] = e^{\lambda b} e^{-\lambda t} g[Y(t)].$$

It follows that $\{e^{-\lambda t} g[Y(t)],\ t \in [0, b]\}$ is a martingale as soon as it is shown that $E\{|g[Y(t)]|\} < \infty$ for each $t \in R^+$. (This is trivial in the case of Theorem 8. 1 since g is bounded.) Letting $b \longrightarrow \infty$, the desired result is then obtained.

Since $Y(t) = \binom{y(t)}{y'(t)}$, we have

$$|g[Y(t)]| \leq \|g\| e^{|y'(t)|},$$

and hence

$$E\{|g[Y(t)]|\} \leq \|g\| E\{e^{|y'(t)|}\}.$$

Thus it suffices to show that $E\{e^{|y'(t)|}\}$ is finite. According to a lemma of Borchers [2, p. 96],

$$E\{e^{|y(t)|}\} \leq K_t e^{|y'(0)|} \tag{8. 3}$$

if $y'(0)$ is a constant random variable, where $K_t = 2\exp\{tK(t)(1+K(t))/2\}$ and $K(t)$ is the constant appearing in hypothesis (B2). In case $y'(0)$ is not a constant random variable, let F be its distribution function. Let $y'_a(t)$ correspond to the constant initial value $y'_a(0, \omega) \equiv a$. Then, for every Borel set G of real numbers,

$$P[y'(t) \in G] = \int_{-\infty}^{\infty} P\{y'_a(t) \in G\} dF(a),$$

and so

$$\begin{aligned} E\{e^{|y'(t)|}\} &= \int_{-\infty}^{\infty} E\{e^{|y'_a(t)|}\} dF(a) \\ &\leq \int_{-\infty}^{\infty} K_t e^{|a|} dF(a) \quad \text{by (8. 3)} \\ &= K_t E\{e^{|y'(0)|}\} < \infty \end{aligned}$$

by hypothesis. This completes the proof.

So we wish to solve one of the equations $Ag = \lambda$ or $Ag = \lambda g$ (or a similar equation with A replaced by $\mathscr{A}$) and thereby obtain information about $\{Y(t),\ t \in R^+\}$ from the fact that a stochastic process closely related to $\{g[Y(t)],\ t \in R^+\}$ is a martingale. If, for instance, $g[Y(t)]$ converges with probability 1 as $t \longrightarrow \infty$, then what can be said about the asymptotic sample function behavior of the y- and y'-processes? It seems that any such analysis depends on knowing the solution function g explicitly.

The next result is a non-existence theorem which shows that Theorem 8. 1 is useless. That is, every solution g in $\mathscr{C}$ of $\mathscr{A}g = \lambda$ or of $\mathscr{A}g = \lambda g$ is identically zero (if $\sigma(x) > 0$ for each $x \in R^2$). On the other hand, we shall later derive some useful consequences of Theorem 8. 2 (see especially Theorem 8. 5).

THEOREM 8. 3. *Let $g \in \mathscr{C}$ be a classical solution of*

$$2^{-1}\sigma^2(\xi, \eta)(\partial^2 g/\partial \eta^2) + m(\xi, \eta)(\partial g/\partial \eta) + \eta(\partial g/\partial \xi) = \begin{cases} \lambda \\ \lambda g \end{cases} \tag{8. 4}$$

where σ *and* m *are continuous and* σ *is positive. Then* $g \equiv 0$.

Note that any solution of $\mathscr{A}g = \lambda$ or of $\mathscr{A}g = \lambda g$, where $\mathscr{A}$ is given by (8. 2), is a classical solution of (8. 4) since $\mathscr{C}$ is equipped with the supremum norm.

Proof. If $g \in \mathscr{C}$ satisfies (8. 4) with right hand side λ, then since $g(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, we must have $\lambda = 0$. Thus it suffices to consider (8. 4) with right hand side λg. Let $\nu = \max(\lambda, 0)$. Let $\delta > 0$ be given. Since g vanishes at ∞, there is a $K = K(\delta) > 0$ such that $|g(\xi, \eta)| \leq \delta$ if $\xi^2 + \eta^2 \geq K$. By the extended form of the Maximum Principle [10, p. 38], we obtain

$$|g(\xi, \eta)| \leq e^{\nu\xi}\delta$$

for $\xi^2 + \eta^2 \leq K$ and hence for all $(\xi, \eta) \in R^2$. Since $\delta > 0$ is arbitrary we must have $g \equiv 0$.

Theorem 8. 2 is thus better suited to the problem of determining asymptotic behavior of the Y-process trajectories than is Theorem 8. 1, since it permits in some cases an explicit computation of a solution of $Ag = 0$ which can "grow at infinity" (we have taken $\lambda = 0$ for convenience). That is to say, in the present case, it is necessary to work with the non-strongly continuous semi-group $\{T_t,\ t \in R^+\}$ rather than with $\{\mathscr{T}_t,\ t \in R^+\}$ in order to get non-trivial results.

The next result deals with the special case $m \equiv 0$.

PROPOSITION 8. 4. *Let* (B1)-(B3) *hold for a second order Itô process described by* (8. 1), *and let* $m \equiv 0$. *Then* $\{y'(t),\ t \in R^+\}$ *is a martingale. Let* $\mathscr{F}_t$ *be the* σ-*field generated by* $\{y(s),\ s \in [0, t]\}$. *If also* (B4) *holds, then*

$$E\{y(t+s)|\mathscr{F}_t\} = y(t) + sy'(t)$$

for all $t,\ s \in R^+$. *Suppose that* $\lim_{t\to\infty} E\{|y'(t)|\}$ *(which exists in* $[0, \infty]$*) is finite. Then, as* $t \longrightarrow \infty$, $y'(t)$ *converges with probability* 1, *say to* $y'(\infty)$, *and*

$$\lim_{t\to\infty}(y(t+s) - y(s)) = sy'(\infty) \tag{8. 5}$$

with probability 1.

Proof. Since $m \equiv 0$,

$$y'(t) - y'(0) = \int_0^t \sigma[s, y(s),\ y'(s)]dz(s),$$

and so $\{y'(t)-y'(0),\ t\in R^+\}$ is a martingale; therefore so is $\{y'(t),\ t\in R^+\}$ a martingale.

If $\mathscr{F}_t$ is as in the proposition, then $\mathscr{F}_t$ is also generated by $\{y'(s), s\in[0,t]\}$; and $\{y'(t),\ \mathscr{F}_t,\ t\in R^+\}$ is a martingale. Let $s,\ t\in R^+$. Then if σ is bounded,

$$\begin{aligned} E\{y(t+s)|\mathscr{F}_t\} &= E\left\{y(t)+\int_t^{t+s} y'(u)du\,|\mathscr{F}_t\right\} \\ &= y(t)+E\left\{\int_t^{t+s} y'(u)du\,|\mathscr{F}_t\right\} \\ &= y(t)+E\left\{\lim_{n\to\infty}\sum_{i=1}^n sn^{-1}y'(t+isn^{-1})|\mathscr{F}_t\right\} \qquad (8.\ 6) \\ &= y(t)+\lim_{n\to\infty}E\{\sum_{i=1}^n sn^{-1}y'(t+isn^{-1})|\mathscr{F}_t\} \qquad (8.\ 7) \end{aligned}$$

(the justification for this interchange is given below)

$$= y(t)+\lim_{n\to\infty}\sum_{i=1}^n sn^{-1}y'(t) = y(t)+sy'(t). \qquad (8.\ 8)$$

(8. 8) follows from (8. 7) since $\{y'(t),\ \mathscr{F}_t,\ t\in R^+\}$ is a martingale. To see why (8. 7) follows from (8. 6) let

$$X = y(t+s)-y(t),\ X_n = \sum_{i=1}^n sn^{-1}y'(t_i)$$

where $t_i = t+isn^{-1}$. It must be shown that the random variable $\lim_{n\to\infty} E\{X_n|\mathscr{F}_t\}$ (which exists and equals $sy'(t)$) equals $E\{X|\mathscr{F}_t\}$. Since

$$\begin{aligned} E\{|E\{X_n|\mathscr{F}_t\}-E\{X|\mathscr{F}_t\}|\} &= E\{|E\{X_n-X|\mathscr{F}_t\}|\} \\ &\leq E\{|X_n-X|\}\leq E^{1/2}\{|X_n-X|^2\}, \end{aligned}$$

it suffices to show that $\lim_{n\to\infty} E\{|X_n-X|^2\}=0$. Now,

$$\begin{aligned} E\{|X_n-X|^2\} &= E\left\{\left|\sum_{i=1}^n sn^{-1}y(t_i)-\int_t^{t+s} y'(u)du\right|^2\right\} \\ &= E\left\{\left|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(y'(t_i)-y'(u))du\right|^2\right\} \\ &\leq nE\left\{\sum_{i=1}^n\left|\int_{t_{i-1}}^{t_i}(y'(t_i)-y'(u))du\right|^2\right\} \\ &\leq nE\left\{sn^{-1}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|y'(t_i)-y'(u)|^2du\right\} \end{aligned}$$

by the Schwarz inequality and the fact that $t_i - t_{i-1} = sn^{-1}$. Consequently

$$E\{|X_n - X|^2\} = s\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} E\{|y'(t_i) - y'(u)|^2\}du. \tag{8. 9}$$

Since $m \equiv 0$, for $v \geq u$, using (8. 1),

$$E\{|y'(v) - y'(u)|^2\} = \mathrm{Var}\,\{y'(v) - y'(u)\} = \int_u^v E\{\sigma^2(y(r),\ y'(r))\}dr$$

(see also (6. 3)). Hence, if K is a bound for σ^2,

$$E\{|y'(v) - y'(u)|^2\} \leq K(v-u).$$

Combining this inequality with (8. 9) we obtain

$$E\{|X_n - X|^2\} \leq sK\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i - u)du$$

$$= sK2^{-1}\sum_{i=1}^{n}(t_i - t_{i-1})^2 = sK2^{-1}\sum_{i=1}^{n}s^2n^{-2} = s^3K/2n \longrightarrow 0$$

as $n \longrightarrow \infty$. Thus (8. 7) follows from (8. 6) when (B4) holds.

If $E\{|y'(t)|\}$, which is a monotone function of t, is bounded, then $\lim_{t\to\infty} y'(t)$ exists with probability 1 by the Martingale Convergence Theorem [4, p. 319]. Let $y'(\infty)$ be the limit random variable. Then for any fixed $s \in R^+$,

$$y(t+s) - y(t) = sy'(r) \longrightarrow sy'(\infty)$$

as $t \longrightarrow \infty$ (by the Mean Value Theorem) with probability 1. The y-process increments thus become "asymptotically stationary".

We remark that (8. 5) is valid even if $E\{|y'(t)|\} \longrightarrow \infty$ as long as $\lim_{t\to\infty} y'(t) = y'(\infty)$ with probability 1. In this case $\{y'(t),\ t \in [0,\infty]\}$ need not be a martingale.

THEOREM 8. 5. *Consider a second order Itô process described by* (8. 1) *for which* (B1)-(B4) *hold with σ positive. Suppose that $p(\eta) = m(\xi,\eta)/\sigma^2(\xi,\eta)$ depends only on η. Assume that m, σ are such that*

(i) *p is bounded*

(ii) *there exists $\eta_0 > 0$ such that p' exists and is bounded and continuous in $\{\eta\colon |\eta| \geq \eta_0\}$.*

Let $g(\xi,\eta) = g(\eta) = \int_0^\eta \exp\left\{-2\int_0^t p(s)ds\right\}dt$. *Then $g \in C_{(2)}$ and $Ag = 0$.* $\{g[y'(t)],$

$t \in R^+\}$ *is a martingale. Suppose that* $g[y'(t)]$ *converges with probability* 1 *as* $t \longrightarrow \infty$. *Then* $y'(t)$ *converges with probability* 1, *say to* $y'(\infty)$, *as* $t \longrightarrow \infty$, *and*

$$\lim_{t\to\infty} (y(t+s) - y(t)) = sy'(\infty)$$

with probability 1 *for each* $s \in R^+$. ($y'(\infty)$ *may take on the values* $\pm\infty$ *with positive probability.*)

Proof. Let the hypotheses hold. A function h will satisfy $Ah = 0$ if and only if

$$(\partial^2 h/\partial\eta^2) + 2p(\eta)(\partial h/\partial\eta) + (2\eta/\sigma^2(\xi,\eta))(\partial h/\partial\xi) = 0.$$

In case $h(\xi,\eta) = h(\eta)$ is independent of ξ, this equation becomes

$$h'' + 2ph' = 0$$

where primes denote differentiation with respect to η. The general solution of this ordinary differential equation is given by

$$C_1 + C_2 \int_0^\eta \exp\left\{-2\int_0^t p(s)ds\right\}dt\,;$$

setting $C_1 = 0$, $C_2 = 1$, it will follow that $Ag = 0$ where

$$g(\xi,\eta) = g(\eta) = \int_0^\eta \exp\left\{-2\int_0^t p(s)ds\right\}dt$$

as soon as it is shown that $g \in C_{(2)}$.

To that end, let $K/4$ be a bound for p, so that $|p(\eta)| \leq K/4$ for all real η. Let $z = K\eta$ and $h(z) = g(z/K)$. Then

$$g''(\eta) + 2p(\eta)g'(\eta) = 0$$

for all real η implies

$$h''(z) + 2r(z)K^{-1}h'(z) = 0$$

for all real z, where $r(z) = p(z/K)$. $|r(z)K^{-1}| \leq 1/4$ for all real z; hence without loss of generality we can (and we shall) assume that $K = 1$. Then

$$|g(\xi,\eta)| \leq \int_0^{|\eta|} e^{t/2}dt = 2(e^{|\eta|/2} - 1) < 2e^{|\eta|/2}$$

for all $(\xi,\eta) \in R^2$. From this and from the formulas

$$g'(\eta) = \exp\left\{-2\int_0^\eta p(s)ds\right\},$$

$$g''(\eta) = -2p(\eta)\exp\left\{-2\int_0^\eta p(s)ds\right\},$$

$$g'''(\eta) = \{-2p'(\eta) + 4p^2(\eta)\}\exp\left\{-2\int_0^\eta p(s)ds\right\},$$

it follows easily using (ii) that $g \in C_{(2)}$ (with $c = 1/2$).

Let

$$h(t) = g[y'(t)] = \int_0^{y'(t)} \exp\left\{-2\int_0^r p(s)ds\right\}dr.$$

It follows by Theorem 8.2 that the stochastic process $\{h(t),\ t \in R^+\}$ is a martingale. Suppose that as $t \longrightarrow \infty$, $h(t)$ converges with probability 1. Then, with probability 1, $y'(t)$ converges as $t \longrightarrow \infty$, since g is a strictly increasing function. Let $y'(\infty) = \lim_{t\to\infty} y'(t)$; $y'(\infty)$ may take on the values $\pm\infty$ with positive probability. For any $s \in R^+$, as in the preceding proposition,

$$\lim_{t\to\infty}(y(t+s) - y(t)) = sy'(\infty)$$

with probability 1, so that the y-process increments become "asymptotically stationary". (In this connection see the example given in [11, p. 85], [6, p. 677].)

Note that by the Martingale Convergence Theorem, $h(t) = g[y'(t)]$ will indeed converge with probability 1 if $E\{|g[y'(t)]|\}$ is bounded. The following result gives a sufficient condition for this to happen.

LEMMA 8.6. *Let p be as in Theorem 8.5. Suppose there exist positive constants γ, C such that $p(s) \geq \gamma$ for $s \geq C$ and $p(s) \leq -\gamma$ for $s \leq -C$. Then the random variables $h(t) = g[y'(t)]$, $t \in R^+$, are uniformly bounded. $\lim_{t\to\infty} h(t)$ exists with probability 1; let $h(\infty)$ denote this limit. Then for every $r \geq 1$,*

$$\lim_{t\to\infty} E\{|h(t) - h(\infty)|^r\} = 0.$$

A simple example of a function p satisfying the hypotheses of this lemma is given by $p(s) = \tan^{-1}s$.

Proof. Let the hypotheses of the lemma hold. Since p is continuous, there is a constant N such that $|p(s)| \leq N$ whenever $s \in [-C, C]$. If $s \geq C$,

$$0 \leq g(s) = \int_0^s \exp\left\{-2\int_0^t p(u)du\right\}dt$$

$$\leq e^{2NC}C + \int_C^s e^{-2\gamma t}dt \leq Ce^{2NC} + (2\gamma)^{-1}e^{-2\gamma s}.$$

Similarly, if $s \leq -C$,

$$0 \geq g(s) \geq -Ce^{2NC} - (2\gamma)^{-1}e^{2\gamma s}.$$

Consequently, for all real s,

$$|g(s)| \leq Ce^{2NC} + (2\gamma)^{-1}e^{-2\gamma|s|} \leq Ce^{2NC} + (2\gamma)^{-1}.$$

Therefore, the random variables $h(t) = g[y'(t)]$, $t \in R^+$, which form a martingale, are uniformly bounded and hence uniformly integrable. Therefore by [4, p. 319], for every $r \geq 1$,

$$\lim_{t\to\infty} E\{|h(t) - h(\infty)|^r\} = 0$$

where $h(\infty)$ is the a.e. limit of $h(t)$ which necessarily exists. Moreover, $\{h(t),\ t \in [0,\infty]\}$ is a martingale and

$$h(t) = g[y'(t)] = E\{h(\infty)|h(s),\ s \in [0,t]\}$$

for each $t \in R^+$.

In Doob's paper [3] the method of stopping times is used in developing a boundary theory for one-dimensional diffusion processes. For higher dimensional diffusions with stationary transition functions, it is known that if a vector process satisfying a certain stochastic equation is "stopped", it becomes the stochastic process described by a new stochastic equation (see [5, p. 354 ff.]). If $\left\{Y(t) = \binom{y(t)}{y'(t)},\ t \in R^+\right\}$ is the vector form of a second order Itô process, and if a stopping time applied to this process yields a new process $\left\{X(t) = \binom{x_1(t)}{x_2(t)},\ t \in R^+\right\}$, the X-process will be the solution process for a stochastic equation, but the x_2-process will not in general be the derivative of the x_1-process. This is one of the reasons why the boundary theory analysis applied in [3] does not extend automatically to the case of a second order Itô process.

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Added in proof. The conjecture made at the end of section 7 is correct. The proof depends on the following result. Let $\{\alpha(t),\ t \in R^+\}$ satisfy (C1)-(C3) of section 4, and let $x(t) = \int_0^t \alpha(s)dz(s)$. Define the *intrinsic time* τ for x by $\tau(t,\omega) = \int_0^t \alpha^2(s,\omega)ds$. Then $\{y(t,\omega) = x(\tau^{-1}(t,\omega),\omega)\colon 0 \leq t < \tau(\infty,\omega)\}$ is a Brownian motion. This theorem is proved in H.P. Mckean, Jr., *Stochastic Integrals*, Academic Press, New York, 1969, pp. 29–31.